# Spaces with strong computable type

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We are interested in topological compact pairs $(\Delta, \Sigma)$, $\Sigma \subseteq \Delta$ which have computable type, i.e. they satisfy the following property: if $X$ is a computably Hausdorff space and $f : \Delta \to X$ is a topological embedding such that $f(\Delta)$ and $f(\Sigma)$ are effective compact sets in $X$, then $f(\Delta)$ is a computable set in $X$. It turns out that the topological properties of the pair play an important role in having/not having computable type: some recent results prove that compact manifolds with boundary, finite topological graphs with boundary and pseudo-$n$-cubes with boundary have computable type, see [1, 2, 3, 4]. Our approach consists in finding new topological techniques to prove more general results.

First, some results have been extended from metric spaces to Hausdorff spaces, but we prove that actually it is equivalent in the following sense: requiring an embedding into a computably Hausdorff space is equivalent to requiring it into a computable metric space or even in the Hilbert cube.

Second, we define a more robust notion which we call strong computable type as follows: A compact metric space $X$ has strong computable type if for every copy $K$ of $X$ in the Hilbert cube, the overt information about $K$ is c.e. relative to the compact information about $K$. We prove that a compact metric space has strong computable type if and only if every copy of it is minimal satisfying some $\Pi_1^0$ property or equivalently some $\Sigma_2^0$ property. As a consequence, given a $\Sigma_2^0$ invariant in the Vietoris or upper Vietoris topology, every minimal element satisfying the invariant has strong computable type. As an example, we define a $\Sigma_2^0$ invariant $E_n$ for each $n$, defined in terms of extension of functions from a compact subset of the Hilbert cube to the $n$-dimensional sphere, and we prove that it covers the results of manifolds and pseudo-$n$-cubes. In fact, it covers more than that: we prove that the surface of revolution of the closed sine curve, which is not a pseudo-$n$-cube, is minimal satisfying $P_2$. In addition, we give a topological characterisation of having strong computable type, which will be helpful to obtain counter-examples.

Furthermore, we investigate the class of compact sets which are locally cones of simplicial complexes. This class includes compact manifolds, finite graphs and any finite simplicial complex. Among them, we characterize the spaces having strong computable type as those whose local neighborhoods satisfy what we call the *surjection* property. The challenge here is to define the right notion of boundary and we propose a notion, which can probably be improved in the near future. Finally, we give some simple counter-examples, i.e. some pairs which do not have strong computable type.

## References

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